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Reynolds Stresses in a Lattice Gas

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I use a previously proposed algorithm, based on Lévy walks, to calculate and discuss longitudinal and transverse velocity correlations in turbulent channel flow. The general approach is that of lattice gas hydrodynamics.

KEY WORDS: Lattice gas hydrodynamics; Lévy walks; turbulent channel flow; velocity–velocity correlations; Reynolds stress tensor.

1. INTRODUCTION

In a previous work[1] I proposed to use an algorithm based on Lévy walks to generate turbulent behavior in lattice gas hydrodynamics. The starting point was a suggestion by Shlesinger *et al.*[2] that Lévy walks furnish the relevant description of enhanced diffusion in fully developed turbulence. One is able in particular to recover Richardson's law,[3] which expresses the fact that the average distance squared between two fluid particles increases as the cube of time. Though one deals with probability density distributions for distances which lack a second-order moment,[2] the formalism of Lévy walks handles these situations through associating a completion time to each distance.[2]

An attractive feature of Lévy walks is the clear physical picture behind them, namely that of momentum mixing over large distance. It is reminiscent of the very successful phenomenological description of turbulent channel flows based on Prandtl's mixing length,[4] and of closure approximations for turbulent flows. Let me recall here that Prandtl's mixing length is not a fixed characteristic number, but that it increases with distance from the channel wall.

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The clear physical picture makes Lévy walks natural candidates for implementation at the microscopic lattice gas level. Since lattice gas hydrodynamics involves particles moving from site to site of a regular lattice, exchanges between sites can be easily implemented through additional algorithmic rules. The implementation is of course not unique when going from the macroscopic level to the microscopic one. What I proposed[1] is a straightforward one, the relevance of which must be tested against experimental results. However, one expects results to be insensitive to the precise form of implementation, provided the basic physics of the approach is respected. The outcome is the equivalent in lattice gas hydrodynamics of a closure approximation to the Navier–Stokes equations.

In ref. 1 I presented and discussed results on flow velocity profiles in turbulent channel flow. In particular in the case when distances are scaled by channel width, and velocity by its maximum value, I showed that the flattening of the velocity profile can correspond to an effective Reynolds number of 10^5 for the right value of z, the exponent of the algebraic probability density distribution of distances [see Eq. (1), Section 2]. I also discussed the logarithmic velocity profile and showed that it had the right magnitude and shape. However, a discussion of Reynolds stresses was lacking. These are the subject of the present work.

The lattice gas model considered is the basic, two-dimensional, hexagonal one without rest particles, and with two-, three-, and four-particle collisions, both symmetric and asymmetric.[5]

In Section 2 I describe the Lévy walk algorithm and the computation of Reynolds stresses, and in Section 3 I present results on the latter. These are followed by a discussion and conclusion.

2. LÉVY WALK ALGORITHM

Let me recall that an update in lattice gas hydrodynamics is a combination of both a translation of particles into the directions of their velocity vectors followed by a momentum and energy-conserving collision, if allowed. In between two successive updates the Lévy walk part of the algorithm proceeds as follows.

Imagine pressure-driven channel flow in the y direction, in a channel of width $2L$ in the x direction. A distance l is drawn from an algebraic probability density distribution of the form

$$p(l) \propto l^{-z} \tag{1}$$

The value of z is 1.35 for the results reported here and the main results of ref. 1. Dependence on z and the fact that it is basically the only parameter are discussed in ref. 1.

Once l is drawn, a point S is uniformly chosen through the channel, and the particle configurations at two sites at equal distances l from S in a direction transverse to the flow, i.e., the x direction, are exchanged. This operation corresponds to momentum exchange over a distance of $2l$, such that overall momentum is indeed conserved.

The algebraic distribution is such that the first moment diverges with the width of the system. This is, however, not the whole story. In Lévy walks a completion time is associated with each distance. Here, for a chosen l, the number of exchanges, or correspondingly the number of points S chosen, thus depends on the value of l. The algorithm is such that the larger l is, the fewer exhanges over distance $2l$ take place. Compared to the update time, this is saying that the characteristic time associated with distance l is larger, the greater l is. This feature is consistent with the corresponding aspect of Lévý walks.[2] Since the typical velocity profile flattening of turbulent channel flow can occur only when momentum exchanges take place over sufficiently large distances, an equivalent statement is that only characteristic jumping times close to the update time play a role. For a given l and the corresponding choice of number of exchanges[1] as many different points S are drawn until the required number of exchanges has taken place.

The two components of the Reynolds stress tensor I am computing are the correlation between longitudinal and transverse fluctuating velocity components and the average of the square of the longitudinal one. More precisely, call u the longitudinal and v the transverse fluctuating velocity components. u is the velocity component in the direction of the flow. Then I am considering $\langle uv \rangle / U^2$ and $\langle u^2 \rangle / U^2$, where U denotes the maximum average velocity, which corresponds to the centerline velocity. u is calculated as the difference of longitudinal velocity components (normalized by the relevant number of particles) between the configurations which are exchanged, and v is related to the transfer of momentum across the (transverse) distance between the two configurations. This entails in particular that the longitudinal–transverse correlation is antisymmetric with respect to the channel center, as it has to be. Results are divided by the length of the channel, and the value calculated is attributed to the bin in which the corresponding midpoint, called S above, lies. (Bins are 10 lattice units wide, for a system of maximum width 640 in lattice units.)

3. RESULTS ON VELOCITY CORRELATIONS

There is one characteristic velocity in turbulent channel flow, which gives the order of magnitude of any velocity correlation. This velocity, called v^*, is determined by the shear stress at the wall. It traditionally

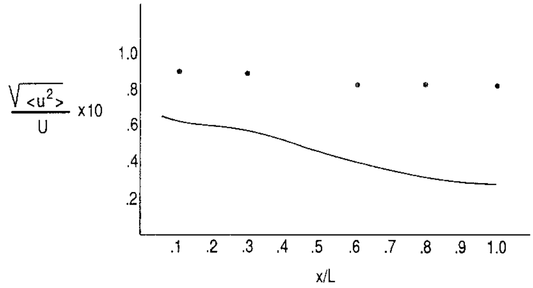

Fig. 1. Longitudinal turbulent velocity correlation, normalized by the maximum average velocity, as a function of distance from the wall normal to the flow. The distance x is normalized by L, which is half the channel width. The wall is at $x = 0$. Numerical results are the points. The comparison is with data from ref. 6, represented by the line. (Very close to the wall the fluctuations go to zero. This region is not shown.)

appears in the scaled variables which enter into the expression of the logarithmic velocity profile. Its value can be extracted from the scaled, flattened velocity profile in the manner described in ref. 4.

The values of $\langle uv \rangle$ and $\langle u^2 \rangle$ therefore must both be of order v^{*2}. This is what is observed experimentally. Another feature which appears in the experimental results is that correlations $\langle uv \rangle$ are smaller than $\langle u^2 \rangle$.

What is therefore expected in our Lévy walk approach?

In ref. 1 it was found for v^* that

$$v^* = 0.02$$

Since the value of v^* is extracted from the scaled velocity profile, it depends on the (only) parameter z in Eq. (1). The value above corresponds

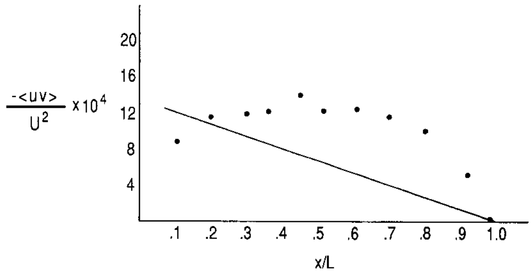

Fig. 2. Negative of the longitudinal–transverse turbulent velocity correlation as a function of distance from the wall across the channel. The wall is at $x = 0$. The normalizations are, respectively, with respect to the square of the maximum average velocity, and the half channel width. Numerical results are the points. The comparison is with data from ref. 6, represented by the line. (Very close to the wall the fluctuations go to zero. This region is not shown.)

to $z = 1.35$, as are all the results reported here. Since we have[1] $U^2 = 0.1$, we should for consistency find that

$$\langle uv \rangle / U^2 \text{ is of order } 4 \times 10^{-3}, \qquad \sqrt{\overline{u^2}}/U \text{ is of order } 6 \times 10^{-2}$$

This is precisely what is found, with the longitudinal fluctuations almost exactly equal to that value, and with longitudinal–transverse fluctuations smaller, as is observed. The results are shown in Figs. 1 and 2, where they are compared with Laufer's[6] results on channel flow at a Reynolds number of 61,600.

4. DISCUSSION AND CONCLUSION

As the comparisons in Figs. 1 and 2 show, the agreement between numerical and experimental results is far from perfect. However, it is, I believe, overall satisfactory, despite discrepancies in magnitude of a factor of two or more for some limited values of wall distance. What is more important than these discrepancies—which I come back to below—is the overall consistency of the Lévy walk approach, which I have stressed in the previous section.

For the longitudinal fluctuations Laufer's[6] data show a stronger dependence on distance, and for longitudinal–transverse correlations the approach to zero at midchannel is not the same. One should not, however, exaggerate the importance of discrepancies, for calculations of correlations involve assumptions. For instance, the length in lattice units over which momentum exchange takes place (relevant to the longitudinal–transverse fluctuations) is to be expressed as a macroscopic length. I have used a basic macroscopic unit of ten lattice gas ones, which corresponds to the width of the domains over which I take time averages, and is also roughly equal to the mean free path for laminar flow. But clearly an uncertainty of a factor of two is possible. There are equally simplifying assumptions in other numerical methods applied to three-dimensional channel[7–9] or turbulent boundary layer flow,[10] and discrepancies with Laufer's results are noted.[7,9] I should point out here that for turbulent boundary layers, where the layer thickness replaces the channel width, results on velocity correlations are similar to those of channel flow.[11]

Dependence of results on Reynolds number has been hitherto neglected. It turns out to be weak for quantities scaled by maximum average velocity and channel width as soon as the Reynolds number is high enough. This is apparent already from the average scaled velocity profile.[4] As shown before[1] for Lévy walks, it is essentially the exponent z in the probability density distribution for distances (cf. Section 2) which controls the dependence on Reynolds numbers: higher values of z lead to a lesser tail for the distribution of distances over which momentum exchanges take place, and consequently there is less flattening of the laminar velocity profile. The average distance involved in exchanges grows weakly for large L as L^{2-z} and therefore dependence on Reynolds number is weak, as indicated by the data.

In conclusion, one can say that the results are encouraging, and that, considering together the results of ref. 1 and those of this work, the Lévy walk approach to turbulent channel flow in lattice gas hydrodynamics leads to a consistent, satisfactory description of experimental results. Many other aspects of turbulent channel flow and of other flows could and need to be investigated within the same approach.

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